# 1 Elementary Euclidean geometry and review of the proof techniques

Mathematics is different from other natural sciences in the way the truth of some conjecture is inferred. While in most cases in natural sciences the ultimate criterion for the truth is an experiment, in mathematics we (mostly) rely on *proof*.

In short, a proof is a convincing argument that some statement is true. Certainly, to make the argument convincing, one needs to stick to the rules of logic and be careful with each step. Since all the students already have taken an intro to proof class, I will not dwell much either on the general discussion on what a proof is, or on the logical rules and laws. Instead, my goal here is to give a few (geometric) examples of proofs. Recall from the same class that one usually distinguishes five different proof techniques: direct proof and its close relative proof by cases, proof of the contrapositive statement and proof by contradiction (these two are called indirect proofs) and finally a proof by mathematical induction. I will save the mathematical induction for future notes and consider here only examples for the first four proof techniques.

## 1.1 Direct proofs

Here I plan to give a few examples of *direct* proofs in Euclidean geometry. By no means I am about to make the way from the basic notions of geometry to more and more complicated results; what follows is just some (relatively) random elementary examples. Therefore it is quite important to keep track of what I assume to be known (to be true) in each proof.

I start with a very basic theorem (recall that *theorem* is a true fact). In each theorem below, in addition to the general statement, I also carefully record what is given specifically, and what is required to be shown (to be proved). It may seem excessive at this point, but my hope is that this additional step may clarify for the students the flow of the logical steps. Finally, I remind that almost no proof is written in its final form from the very beginning; it is (almost always) advisable to make some scratch work first, often go in the backward direction: from what to be shown eventually to the point where I need to start; I am not showing this step here, presenting just the final result.

**Theorem 1.1.** *If two lines intersect, the vertically opposite angles are equal.*

*Given:* Two straight lines that intersect at the point $O$. The angles $\alpha$ and $\beta$ are vertically opposite (see Fig. 1).

*Prove:* $\alpha = \beta$.

*Proof.* Any two intersecting lines form two pairs of opposite angles, hence without loss of generality we can consider the case as in Fig. 1. In Fig. 1 it is seen that $\alpha + \gamma$ and $\gamma + \beta$ are the angle measures of a flat angle. Since all the angle measures of flat angles are the same, I have

$$\alpha + \gamma = \gamma + \beta,$$

or

$$\alpha = \beta$$

as required. ∎

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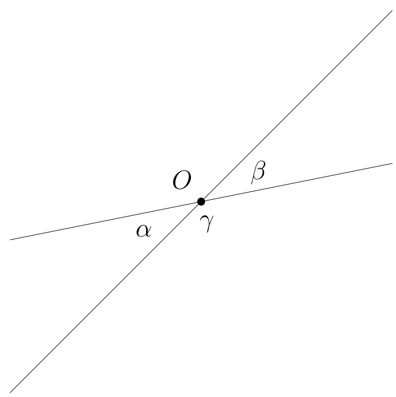


Figure 1: Drawing for Theorem 1.1.

**Remark 1.2.** The only general fact that I used in the proof above is the (quite obvious) fact that the angle measure of a flat angle (i.e., of any of two angles between two rays of a straight line divided by a point) is the same for any straight line, nothing else.

As the next example let me prove the following well known theorem (I will discuss it again when we start talking about Euclid's *Elements*).

**Theorem 1.3.** *The base angles of an isosceles triangle are equal.*

*Given:* Triangle $\triangle ABC$ such that $AB = BC$.
*Prove:* $\angle BAC = \angle BCA$.

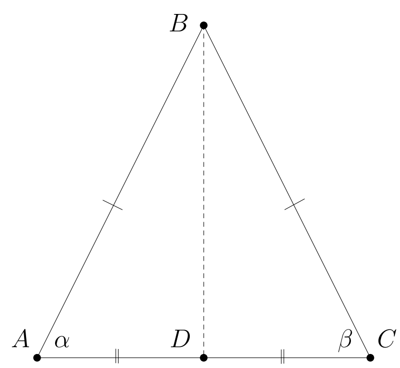


Figure 2: Drawing for Theorem 1.3.

*Proof.* Here and in the proofs below I will usually use Greek letters to denote angles, so $\alpha = \angle BAC$, $\beta = \angle BCA$.

Let $D$ be the middle point of the side $AC$, $AD = DC$. Consider $\triangle ABD$ and $\triangle BCD$. I have

$$\begin{aligned} AB &= BC && \text{(by the given condition)} \\ BD &= BD && \text{(commond side)} \\ AD &= DC && \text{(by construction)} \end{aligned}$$

Therefore in $\triangle ABD$ and $\triangle BCD$ three sides are equal, and therefore, by the side–side–side comparison these two triangles are congruent, and therefore the corresponding angles are equal. Hence

$$\alpha = \beta.$$

$\blacksquare$

**Remark 1.4.** Here I used several facts, some are more obvious than the others. First, quite naturally, I assumed that I always can find the middle of a given segment. Second, much less straightforward, I assumed that I already know at least one sufficient condition for two triangles to be congruent (recall that two figures are congruent if the have the same shape and size). That is, I assumed as a true fact that if three pairs of sides of two triangles are equal in length, then the triangles are congruent (there are two more comparisons, recall them).

Note also that I proved more than it was asked. As a side result I found that $\angle ABD = \angle DBC$, and hence the segment $BD$ is not only the median (by construction) but also a bisect of $\angle ABC$. Assuming that I know that the total angle measure of a triangle is equal to the angle measure of the flat angle (i.e., $\pi$ radian), then it is almost immediate (do it) to show that $\angle ADB = \angle BDC = \pi/2$, i.e., $AD$ is also a height of $\triangle ABC$. That is, in an isosceles triangle the median to the base is also the bisect and the height.

**Exercise 1.** Recall that for logical statement $P \implies Q$ its *converse* is the statement $Q \implies P$. These are not logically equivalent, that is if we know that one is true we usually cannot say anything about the truth of the other. Having said this, in mathematics it is very often important, after proving some theorem of the form $P \implies Q$, to ask what can be said about the converse statement. If it turns out that the converse is also true we say that $P$ and $Q$ are equivalent, which is often denoted as $P \Leftrightarrow Q$; other ways to say the same are: $P$ if and only if $Q$, or $P$ is necessary and sufficient for $Q$.

It turns out that the converse statement to Theorem 1.3 is true. That is, in this exercise I ask you to come up with a proof that if in $\triangle ABC$ $\angle BAC = \angle BCA$ then the sides opposite to these angles are equal, and hence proving

**Theorem 1.5.** *A triangle is isosceles if and only if it has two equal angles.*

Now I would like to consider a variation of the direct proof: *proof by cases.* The logical structure is the same, the difference here is that I divide my proof into several cases, the most important point is not to miss any (however small) case.

**Theorem 1.6.** *The inscribed angle is equal to the half of the angle measure of the arc on which it sits.*

*Given:* A circle with the center $O$ and point $A$ on the circle, such that two rays from point $A$ intersect the circle at points $B$ and $C$.

*Prove:* $\angle BAC = \frac{1}{2}\angle BOC$, where $\angle BOC$ is chosen such that it corresponds to the arc of the circle from $B$ to $C$ that does not contain $A$.

*Proof.* It is possible to have three cases: ($a$) one of the rays from $A$ passes through $O$, ($b$) point $O$ is between the rays from $A$, and ($c$) point $O$ is not between the rays from $A$. Consider these cases one by one.

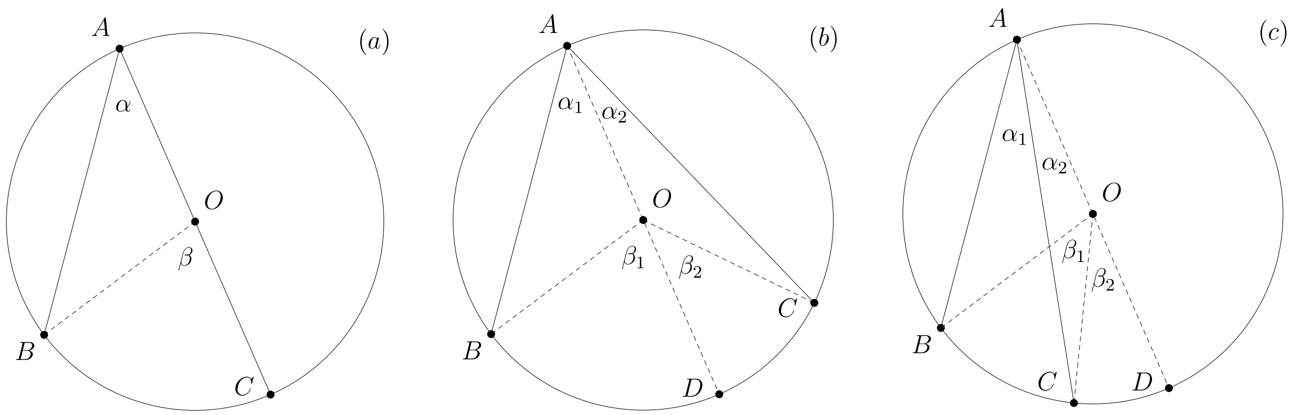


Figure 3: Drawing for the three cases of Theorem 1.6.

Case $(a)$. In Fig. 3$(a)$ one can see that in $\triangle AOB$ $AO = OB$ (they are both radii of the circle), hence $\angle ABO = \alpha$ by Theorem 1.3. $\angle AOB$ and $\beta$ are supplementary, i.e., $\angle AOB = \pi - \beta$. Since the sum of the angles of any triangle is $\pi$, I get

$$\alpha + \alpha + \pi - \beta = \pi,$$

or, after simplification,

$$\alpha = \frac{1}{2}\beta$$

as claimed. Case $(a)$ has been proven.

Case $(b)$. See Fig. 3$(b)$. Since $O$ is between the rays $AB$ and $AC$ then there is point $D$, such that $AD$ passes through $O$. From Case $(a)$ I have that $\alpha_1 = \frac{1}{2}\beta_1$, $\alpha_2 = \frac{1}{2}\beta_2$, which implies

$$\alpha_1 + \alpha_2 = \frac{1}{2}(\beta_1 + \beta_2),$$

or

$$\angle BAC = \frac{1}{2}\angle BOC.$$

Case $(b)$ has been proven.

Case $(c)$. This case is similar to the previous. Note (see the figure) that

$$\angle BAD = \frac{1}{2}\angle BOD, \quad \alpha_2 = \frac{1}{2}\beta_2,$$

using again Case $(a)$. This implies that

$$\alpha_1 = \angle BAD - \alpha_2 = \frac{1}{2}(\angle BOD - \beta_2) = \frac{1}{2}\beta_2,$$

as required. Theorem has been proven. ■

**Remark 1.7.** The key fact that I assumed to be known in the proof above is that the sum of the angles of any triangles is equal to $\pi$. I also relied on the proven before Theorem 1.3.

Theorem 1.6 has a few immediate corollaries whose proofs are left as exercises. Recall that a polygon is called *cyclic* if it can be inscribed into a circle. For instance any triangle is cyclic (can you prove it?) but not every quadrilateral is cyclic (give an example).

**Corollary 1.8** (Thales's theorem)**.** *An angle inscribed in a semicircle is the right angle (Fig. 4(a)).*

**Corollary 1.9.** *Two inscribed angles are equal if they sit on the same arc (Fig. 4(b)).*

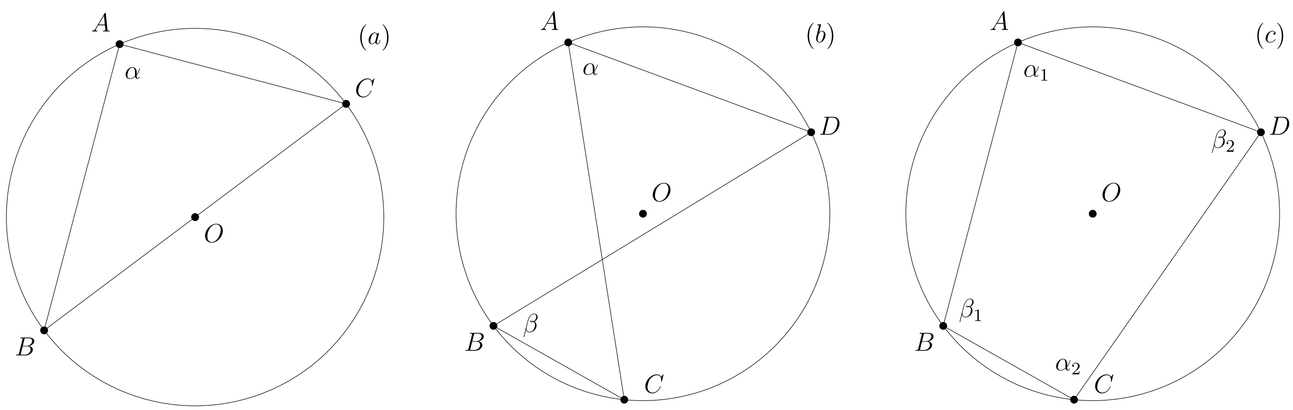


Figure 4: Drawing for Corollaries to Theorem 1.6.

**Corollary 1.10.** *In a cyclic quadrilateral the opposite angles are supplementary (Fig. 4(c)) (i.e., they sum to $\pi$).*

**Exercise 2.** It turns out that the last two corollaries have true converses. Putting everything together I encourage the reader to prove

**Theorem 1.11.** *A quadrilateral is cyclic if and only if the angles between one side and diagonal is equal to the angle between the opposite side and other diagonal.*

and

**Theorem 1.12.** *A quadrilateral is cyclic if and only if its opposite angles are supplementary.*

## 1.2 Indirect proofs

There are two techniques of indirect proofs: contrapositive and proof by contradiction. Let me start with the former one. Recall that the contrapositive statement for $P \implies Q$ is not $Q \implies$ not $P$, and these two statements are logically equivalent, i.e., if we know that one is true then automatically the other one is true. Therefore sometimes it is convenient to replace the original statement of a theorem with its contrapositive, prove it, and hence prove the original statement as well. A good example of this approach is given by the Steiner–Lehmus theorem (I encourage the reader to read more about this theorem online; the only thing that I would like to mention that the indirect proof of this theorem turned out much simpler than any direct approach).

**Theorem 1.13** (Steiner–Lehmus)**.** *A triangle with two angle bisects of equal length is isosceles.*

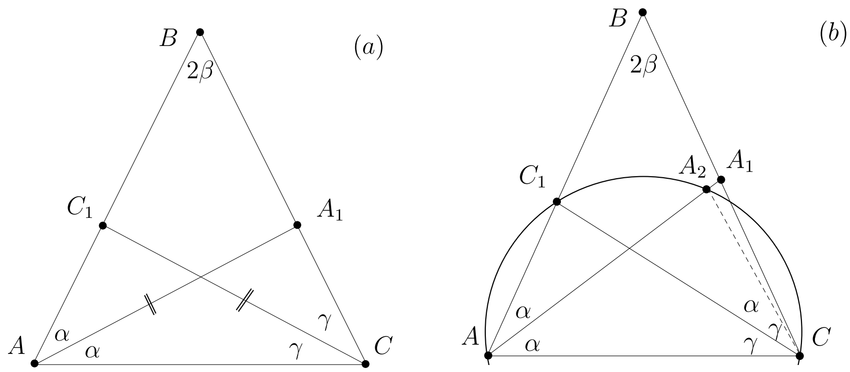


Figure 5: Drawing for Theorem 1.13.

*Given:* A triangle $\triangle ABC$ with the bisects $AA_1$ and $CC_1$ to the angles $\angle BAC$ and $\angle BCA$ respectively (Fig. 5($a$)). $AA_1 = CC_1$.

*Prove:* $AB = BC$.

*Proof.* To prove this theorem I consider the contrapositive statements: If two sides $AB$ and $BC$ are not equal in length then the bisects to the angles opposite to these sides are also not equal in length (i.e., $AA_1 \neq CC_1$, see Fig. 5($b$)).

Since the triangle is not isosceles, the angles at the base are not equal (Theorem 1.5). Therefore $2\alpha \neq 2\gamma$, and hence, without loss of generality, I assume that

$$\alpha < \gamma.$$

Moreover, from the fact that $2\alpha + 2\beta + 2\gamma = \pi$ it follows that $\alpha + \beta + \gamma = \frac{\pi}{2}$, and hence $\alpha + \gamma$ is an acute angle (I will need it below).

Now, since $\alpha < \gamma$, I can find point $A_2$ on $AA_1$ such that $\angle C_1CA_2 = \alpha$ (Fig. 5($b$)).

Consider the quadrilateral $AC_1A_2C$. By Theorem 1.11 it is cyclic ($\angle A_2AC_1 = \angle A_2CC_1$) and hence I can build a circle passing through these points. Note that $AC_1A_2$ is the arc that corresponds to the angle $\alpha + \gamma$, and $C_1A_2C$ is the arc that corresponds to the angle $2\alpha < \alpha + \gamma$. $AA_2$ and $C_1C$ are the corresponding chords. Since both $2\alpha$ and $\alpha + \gamma$ are acute angles, and $2\alpha < \alpha + \gamma$ hence the chords satisfy the relation $AA_2 > CC_1$. Finally since $AA_1 > AA_2$ (Fig. 5($b$)) I conclude that

$$AA_1 > CC_1$$

as required. ∎

**Remark 1.14.** Let me collect together the facts that I assumed to be true in the proof above. First, I used Theorems 1.5 and 1.11, which were assigned as exercises. I also used the fact (make a sketch) that if I have two acute inscribed angles $\alpha < \beta$ then the corresponding chords on which they sit satisfy the relation $\text{chord}_\alpha < \text{chord}_\beta$, which seems to be very plausible but still requires proof.

**Exercise 3.** Probably not surprisingly the converse to Theorem 1.13 is also true. I invite the reader to provide a proof of it (it should be much easier and does not require anything beyond what has been discussed so far in this note).

Another technique of an indirect proof is the so-called *proof by contradiction.* I will consider arguably the most famous theorem that is proved by contradiction, but instead of showing the usual algebraic proof, I will use some basic geometric constructions.

**Theorem 1.15.** *$\sqrt{2}$ is not rational.*

*Proof.* As usual I make an assumption that $\sqrt{2}$ is rational. This means that there are two natural numbers $p$ and $q$, $p > q$ such that the ratio of their squares is equal to 2:

$$\frac{p^2}{q^2} = 2.$$

Since $p^2$ and $q^2$ are the areas of squares with the sides $p$ and $q$ respectively, my assumption is equivalent to the existence of two squares with the natural sides $p$ and $q$ (Fig. 6($a$)). It is clear that if there is one then there exist infinitely many such pairs (I can, say, multiply both $p$ and $q$ by 2 and get another pair, and so on). However, since the set of natural numbers is bounded from below (there is no natural number smaller than 1), then it must be true that there exists a pair of such squares with the smallest possible $p$ and $q$. So I assume that my pair of squares is the smallest one.

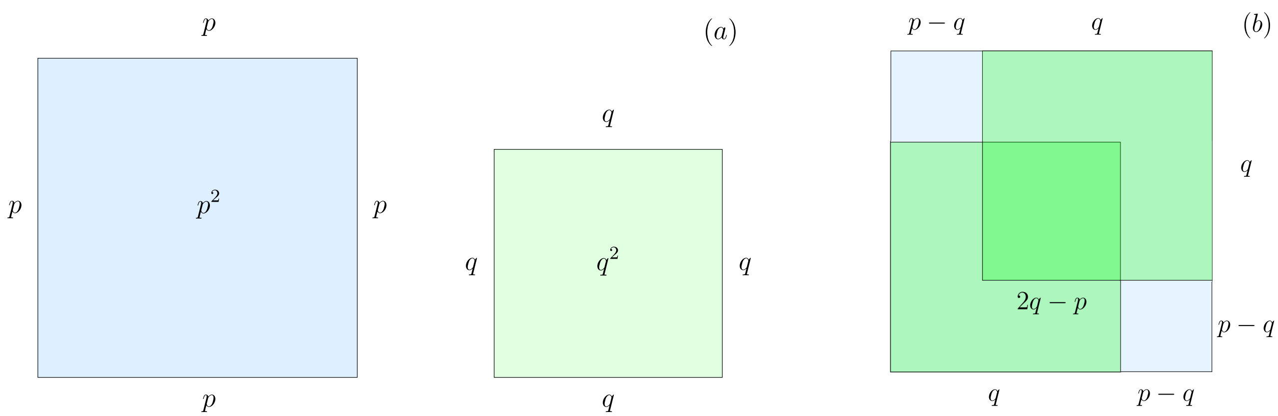


Figure 6: Two squares with integer sides whose ratio of areas is equal to 2. ($a$) Separately. ($b$) Two copies of the smaller square inside the larger one.

Now let me put the right square from Fig 6($a$) inside the bottom left and top right corners of the left square (Fig. 6($b$)). Since $p, q$ are natural then $p-q$ and $2q-p$ are also natural numbers, and moreover these are the numbers that correspond to the sides of three squares I observe in Fig. 6($b$). Even further, since the area of the bigger square is exactly twice the area of the square with side $q$ (light green), therefore, from geometric considerations, the area of the square with the side $2q-p$ is exactly equal the sum of the areas of the squares withe the side $p-q$, or twice the area of one square with the side $p-q$. In other words, I found another pair of squares with integer sides, the ratio of areas of which is 2, and clearly these squares are smaller than the ones I started with. Since I also assumed that I started with the smallest possible pair of such squares, I have reached a contradiction, and therefore my initial statement that there exists such a pair of squares cannot be true. Theorem has been proven. ■

**Exercise 4.** Find a geometric proof that $\sqrt{3}$ is irrational. *Hint:* Consider two equilateral triangles with integer sides $p$ and $q$ whose area ratio is equal to 3.

**Remark 1.16.** In my proof above I relied on some relatively obvious geometric constructions about squares, and also, quite significantly, on the properties of natural numbers.

**Remark 1.17.** Proofs by considering contrapositive and by contradiction are often mixed by the students. I encourage the reader to carefully analyze the logical structure of Theorems 1.13 and 1.15. In the former case I considered the statement $\text{not}\, Q \implies \text{not}\, P$, and filled the implication sign with a number of convincing direct arguments. In the latter case, in the proof by contradiction, I usually start with premises $P$ and $\text{not}\, Q$ (I need both!), and in the process of my proof I need to establish that for some additional statement $R$ both $R$ and $\text{not}\, R$ are true, thus reaching the contradiction. In Theorem 1.15 such statement $R$ was "The squares with the integer sides $p$ and $q$ are the smallest possible." Think this out.

Up till now all the examples I consider did not constitute many logical steps. To finish this section I would like to give an example of a somewhat more involved geometric proof.

**Theorem 1.18** (Heron's formula)**.** *The area $S$ of a triangle with sides $a, b$ and $c$ can be computed as*

$$S = \sqrt{s(s-a)(s-b)(s-c)}\,,$$

*where*

$$s = \frac{a+b+c}{2}$$

*is the half perimeter of this triangle.*

*Proof.* ∎